Gelfand-Shilov type spaces through Hermite expansions

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Abstract. Gelfand-Shilov spaces of the type $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ and $\sum_\alpha^\alpha(\mathbb{R}^d)$ can be realized as sequence spaces by means of the Hermite representation Theorem. In this article we show that for a function $f = \sum a_k \Psi_k \in \mathcal{S}_\beta^\beta(\mathbb{R}^d)$ (resp. $\sum_\beta^\beta(\mathbb{R}^d)$) in order that $\sum a_k \Psi_k \to f$ in $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$), where $\frac{1}{2} \leq \alpha \leq \beta$ (resp. $1/2 < \alpha \leq \beta$), it follows that $\alpha = \beta$. Furthermore we characterize spaces of the type $\big(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta\big)(\mathbb{R}^{s+t})$ (resp. $\big(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta\big)(\mathbb{R}^{s+t})$) defined by Gelfand and Shilowthrough the estimmates Hermite coefficients and, moreover, introduce a new spaces of Gelfand-Shilov type $S_\sigma^{\otimes,\sigma}(\mathbb{R}^n), \sigma \geq 1/2$, and $\Sigma_\sigma^{\otimes,\sigma}(\mathbb{R}^n), \sigma > 1/2$. All the spaces are compared.

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1. Introduction

Gelfand-Shilov spaces $\mathcal{S}_\alpha(\mathbb{R}^d), \mathcal{S}^\beta(\mathbb{R}^d)$ and $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ and their generalisations, the Gelfand-Shilov spaces of Roumieau and Beuerling type $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ respectivly $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ are discussed in [2], [3], [4], [6], [7], [9], [10], [11] and [12]. In this paper we focus on the special cases $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$, resp., $\sum_\alpha^\beta(\mathbb{R}^d)$. We show that if the Hermite expansion $\sum a_k \Psi_k$ converges to f (here Ψ_k denote the d-dimensional Hermite functions and a_k the Hermite coefficients of f) in the sense of $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ $(\alpha < \beta)$, resp., $\sum_\alpha^\beta(\mathbb{R}^d)$ $(\alpha < \beta)$, then it belongs to $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$, resp., $\sum_\alpha^\alpha(\mathbb{R}^d)$.
Furthermore we analyze intermideate spaces $\big(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta\big)(\mathbb{R}^{s+t})$ and $\big(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta\big)(\mathbb{R}^{s+t})$, introduced also by Gelfand and Shilov, through the estimates of Hermite coefficients. The elements of spaces of this type are functions f which behave in their first s components like a function in $\mathcal{S}_\alpha^\alpha(\mathbb{R}^s)$ (resp. $\sum_\alpha^\alpha(\mathbb{R}^s)$) and in their last t

components like a function in $\mathcal{S}_\beta^\beta(\mathbb{R}^t)$ (resp. $\sum_\beta^\beta(\mathbb{R}^t)$). In the last part of the paper we introduce one more class of Gelfand-Shilov type spaces $S_\sigma^{\otimes,\sigma}(\mathbb{R}^n), \sigma \geq 1/2$, and $\Sigma_\sigma^{\otimes,\sigma}(\mathbb{R}^n), \sigma > 1/2$. These spaces were obtained through the iteration of Harmonic oscilators and are related to our study of Weyl formula for tensorised products of elliptic Shubin type operators (see [1]). We compare all the considered spaces through the estimates of Hermite coefficients.

1.1. Notation and basic notions

In this paper we use the following notation: Let $j,p,q \in \mathbb{N}_0^d$ and $\alpha, \beta \in \mathbb{R}_+$, we have $p^{p\alpha} = p_1^{p_1\alpha} \cdot \ldots \cdot p_d^{p_d\alpha}$ and the analogoues for $q^{q\beta}$. Similarly for $x \in \mathbb{R}^d$ we have $x^p = x_1^{p_1} \cdot \ldots \cdot x_d^{p_d}$ and for $c \in \mathbb{R}$ we have $c^j = c^{j_1} \cdot \ldots \cdot c^{j_d}$. In addition we write $\partial^q f = \frac{\partial^{|q|}}{\partial_{x_1}^{q_1} \ldots \partial_{x_d}^{q_d}} f$ for $f \in \mathcal{C}^\infty(\mathbb{R}^d)$.
We want to recall a few definitions and facts corresponding to the spaces of type $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ and $\sum_\alpha^\beta(\mathbb{R}^d)$.
The Gelfand-Shilov spaces are defined as follows, cf. [10, Theorem 2.6]:

Definition 1.1. Let $\alpha, \beta \in \mathbb{R}_+$, $p,q \in \mathbb{N}_0^d$ and assume that A, B, C are positive numbers.

1. The Gelfand-Shilov space of type $\mathcal{S}_{\alpha,A}(\mathbb{R}^d)$ is defined as follows:

$$\mathcal{S}_{\alpha,A}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) \ : \ \forall q \ \exists \, C_q \text{ s.t. } \|x^p \partial^q f\|_{L^2(\mathbb{R}^d)} \leq C_q A^{|p|} p!^\alpha \text{ for all } p \right\}.$$

2. The Gelfand-Shilov space of type $\mathcal{S}^{\beta,B}(\mathbb{R}^d)$ is defined as follows:

$$\mathcal{S}^{\beta,B}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) \ : \ \forall p \ \exists \, C_p \text{ s.t. } \|x^p \partial^q f\|_{L^2(\mathbb{R}^d)} \leq C_p B^{|q|} q!^\beta \text{ for all } q \right\}.$$

3. The Gelfand-Shilov space of type $\mathcal{S}_{\alpha,A}^{\beta,B}(\mathbb{R}^d)$ is defined as follows:

$$\mathcal{S}_{\alpha,A}^{\beta,B}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) \ : \ \exists \, C \text{ s.t. } \forall p,q \ : \ \|x^p \partial^q f\|_{L^2(\mathbb{R}^d)} \leq C A^{|p|} p!^\alpha B^{|q|} q!^\beta \right\}.$$

Their inductive and projective limits are denoted by:

1. $\mathcal{S}_\alpha(\mathbb{R}^d) = \text{indlim}_A \mathcal{S}_{\alpha,A}(\mathbb{R}^d)$; $\quad \sum_\alpha(\mathbb{R}^d) = \text{projlim}_A \mathcal{S}_{\alpha,A}(\mathbb{R}^d)$
2. $\mathcal{S}^\beta(\mathbb{R}^d) = \text{indlim}_B \mathcal{S}^{\beta,B}(\mathbb{R}^d)$; $\quad \sum^\beta(\mathbb{R}^d) = \text{projlim}_A \mathcal{S}^{\beta,B}(\mathbb{R}^d)$
3. $\mathcal{S}_\alpha^\beta(\mathbb{R}^d) = \text{indlim}_{A,B} \mathcal{S}_{\alpha,A}^{\beta,B}(\mathbb{R}^d)$; $\sum_\alpha^\beta(\mathbb{R}^d) = \text{projlim}_{A,B} \mathcal{S}_{\alpha,A}^{\beta,B}(\mathbb{R}^d)$

These spaces are subspaces of the Schwartz space

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) \ : \ \forall p,q \ : \ \|x^p \partial^q f\|_{L^2(\mathbb{R}^d)} < \infty \right\}.$$

Note that the space $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ is nontrivial if $\alpha + \beta \geq 1$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$ is nontrivial if $\alpha + \beta > 1$) and that holds $\mathcal{S}_\alpha^\beta(\mathbb{R}^d) \subseteq \mathcal{S}_{\alpha'}^{\beta'}(\mathbb{R}^d)$ and $\sum_\alpha^\beta(\mathbb{R}^d) \subseteq \sum_{\alpha'}^{\beta'}(\mathbb{R}^d)$ if $\alpha \leq \alpha'$ and $\beta \leq \beta'$.

The polynomials

$$H_k(t) = (-1)^k \exp\left(t^2\right) \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \exp\left(-t^2\right), \ t \in \mathbb{R}, \ k \in \mathbb{N}_0$$

are the Hermite polynomials. The one-dimensional Hermite functions ψ_k are given by

$$\psi_k(t) = \left(\pi^{\frac{1}{2}} 2^k k!\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}t^2\right) H_k(t),\ t \in \mathbb{R},\ k \in \mathbb{N}_0.$$

The d-dimensional Hermite functions are

$$\Psi_m(x) = \prod_{j=1}^{d} \psi_{m_j}(x_j) \text{ with } x = (x_1, \ldots, x_d) \in \mathbb{R}^d \text{ and } m \in \mathbb{N}_0^d.$$

For $f \in \mathbb{S}(\mathbb{R}^d)$ the Hermite coefficients are

$$(a_m)_{m \in \mathbb{N}_0^d} = \Big((f, \Psi_m)_{L^2(\mathbb{R}^d)}\Big)_{m \in \mathbb{N}_0^d}. \tag{1.1}$$

Lemma 1.2. *Let $\alpha, \beta \in \mathbb{R}_+$. Then the following holds:*

$$\mathcal{S}_\alpha(\mathbb{R}^d) \cap \mathcal{S}^\beta(\mathbb{R}^d) = \mathcal{S}_\alpha^\beta(\mathbb{R}^d)$$

This result was obtained by Kashpirovskii [5] and later by other authors, cf [2]. In addition it holds $\sum_\alpha^\beta(\mathbb{R}^d) = \sum_\alpha(\mathbb{R}^d) \cap \sum^\beta(\mathbb{R}^d)$ cf [2, Theorem 7.2.2].
We define the spaces of the Hermite coefficients of $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ (resp. $\sum_\alpha^\alpha(\mathbb{R}^d)$) as follows:

$$\mathbf{s}_\alpha^\alpha = \left\{ (a_n)_{n \in \mathbb{N}_0^d} \text{ there exists } t > 0 : \sum_{n \in \mathbb{N}_0^d} |a_n|^2 \exp\left(t\, |n|^{\frac{1}{2\alpha}}\right) < \infty \right\},$$

$$\tilde{\mathbf{s}}_\alpha^\alpha = \left\{ (a_n)_{n \in \mathbb{N}_0^d} \text{ for all } t > 0 : \sum_{n \in \mathbb{N}_0^d} |a_n|^2 \exp\left(t\, |n|^{\frac{1}{2\alpha}}\right) < \infty \right\}.$$

The following lemma was proven in [2] and [7].

Lemma 1.3. *Let $\alpha \geq \frac{1}{2}$ (resp. $\alpha > \frac{1}{2}$). The mapping between $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ (resp. $\sum_\alpha^\alpha(\mathbb{R}^d)$) and the space of the Hermite coefficients, denoted $\mathbf{s}_\alpha^\alpha$ (resp. $\tilde{\mathbf{s}}_\alpha^\alpha$), $f = \sum_{m \in \mathbb{N}_0^d} a_m \Psi_m \to (a_m)_{m \in \mathbb{N}_0^d}$, is a topolocical isomorphism.*

2. The Hermite representation of $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ and $\sum_\alpha^\beta(\mathbb{R}^d)$

Let $\alpha, \beta \in \mathbb{R}_+$ and $\frac{1}{2} \leq \alpha < \beta$ (resp. $\frac{1}{2} < \alpha < \beta$). Let X the set of all functions $f \in \mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$) s.t. their series expansion $f = \sum\limits_{k \in \mathbb{N}_0^d} a_k \Psi_k$ within $\mathcal{S}_\beta^\beta(\mathbb{R}^d)$ (resp. $\sum_\beta^\beta(\mathbb{R}^d)$) converges in the sense of the topology of $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$).

Theorem 2.1. *If for every $f = \sum a_k \Psi_k \in X$, there exist positive constants C_f, c_f and s_f s.t.*

$$|a_k| \leq C_f \exp\left(-c_f\, |k|^{s_f}\right), \ \textit{for } k \in \mathbb{N}_0^d \textit{ or resp.} \tag{2.1}$$

$$|a_k| \leq C_f \exp\left(-t\, |k|^{s_f}\right), \ \textit{for every } t > 0, \ \textit{for } k \in \mathbb{N}_0^d \tag{2.2}$$

then (2.1) *(resp.* (2.2)*) is true for*

$$s_f = \frac{1}{2\alpha}.$$

In particular, this implies $X = \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ (resp. $X = \sum_\alpha^\alpha(\mathbb{R}^d)$).

Proof. For the sake of simplicity we will only prove the case of $X \subset \mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ and $d = 1$. But the proof in the case $X \subset \sum_\alpha^\beta(\mathbb{R}^d)$ is analogous just like the cases of higher dimensions.

Let $f \in X$, i.e. f has a convergent Hermite expansion in $\mathcal{S}_\alpha^\beta(\mathbb{R})$. This implies by lemma 1.2, that f has a convergent Hermite expansion in the spaces $\mathcal{S}_\alpha(\mathbb{R})$ and $\mathcal{S}^\beta(\mathbb{R})$. Then the sequence $\left(\sum_{k=0}^N a_k \psi_k\right)_{N\in\mathbb{N}}$ of partial sums is Cauchy in $\mathcal{S}_\alpha(\mathbb{R})$ and $\mathcal{S}^\beta(\mathbb{R})$, i.e. for $A > 0$:

a) $\frac{\|x^p \sum_{k=N}^M a_k\psi_k\|_{L^2(\mathbb{R})}}{A^p p!^\alpha} \to 0$ for all p if $N, M \to \infty$ and

b) $\frac{\|\partial^q \sum_{k=N}^M a_k\psi_k\|_{L^2(\mathbb{R})}}{A^q q!^\beta} \to 0$ for all q if $N, M \to \infty$.

It is known (cf. [8, 1.1]) that $\partial^q \psi_k = (\mathrm{i}x)^q\, \mathcal{F}\psi_k = (\mathrm{i}x)^q\, (-\mathrm{i})^k\, \psi_k$, where $\mathcal{F}$ is the Fourier transform. Thus we have

$$\|\partial^q \sum_{k=N}^M a_k\psi_k\|_{L^2(\mathbb{R})} = \sum_{k=N}^M \|a_k x^q \psi_k\|_{L^2(\mathbb{R})}.$$

Therefore we only have to consider case a). Note, by a) with $M = N$, for every $\varepsilon > 0$ there exists $N_0(\varepsilon)$ such that:

$$\frac{\|x^p a_N \psi_N\|_{L^2(\mathbb{R})}}{A^p p!^\alpha} < \varepsilon \text{ for } N \geq N_0(\varepsilon) \tag{2.3}$$

uniformly in $p \in \mathbb{N}_0$. We use the well known fact ([10, eq. 1.8]), that

$$x\psi_k(x) = \sqrt{\frac{k}{2}}\psi_{k-1}(x) + \sqrt{\frac{k+1}{2}}\psi_{k+1}(x)$$

and by the use of L^2-norm and Parseval's formula, we obtain by induction that for all $k, p \in \mathbb{N}_0$

$$\|x^p\psi_k\|_{L^2(\mathbb{R})} \geq C^{p+1} k^{\frac{p}{2}} \tag{2.4}$$

for suitable $C > 0$. Now suppose that the assertion is not true which implies that there exists a function $f = \sum_{k=0}^\infty a_k\psi_k \in X$ such that, for a subsequence

$\big(a_{k_j}\big)_{k_j\in\mathbb{N}_0}$ of $\big(a_k\big)_{k\in\mathbb{N}_0}$, it holds

$$\big|a_{k_j}\big| = C_f \exp\Big(-c_f r(k_j) k_j^{\frac{1}{2\alpha}}\Big) \text{ with } j \in \mathbb{N}_0$$

where $\big(r(k_j)\big)_{k_j\in\mathbb{N}_0}$ is a sequence of positive numbers not bounded from below by a $c > 0$. Reformulating the quoted facts, if (2.1) does not hold, then there exists a function $f = \sum_{k=0}^{\infty} a_k\psi_k \in X$ such that

$$\big|a_k\big| = C_f \exp\Big(-c_f r(k) k^{\frac{1}{2\alpha}}\Big) \text{ with } k \in \mathbb{N}_0 \tag{2.5}$$

and

$$r(k) \to 0 \text{ as } k \to \infty.$$

Therefore, (2.3) and (2.4) give, for large enough N,

$$\varepsilon > \frac{\|x^p a_N \psi_N\|_{L^2(\mathbb{R})}}{A^p p!^\alpha} \geq \frac{C_f \exp\big(-c_f r(N) N^{1/2\alpha}\big) C^p N^{p/2}}{A^p p!^\alpha} \tag{2.6}$$

uniformly in $p \in \mathbb{N}_0$. Now we use the inequality

$$\sup_{p\in\mathbb{N}_0} \left(\frac{C^2 N^{1/2}}{Ap!^{\alpha/p}}\right)^p \geq \exp\big(HN^{\frac{1}{2\alpha}}\big) \tag{2.7}$$

which holds for suitable $H > 0$. Thus (2.6) and (2.7) imply that for N large enough

$$\varepsilon > C_f \exp\Big(\big(-c_f r(N) + H\big) N^{\frac{1}{2\alpha}}\Big)$$

and this is not true since $(-c_f r(N) + H) N^{\frac{1}{2\alpha}} \to \infty$ $N \to \infty$. This completes the proof. □

3. Tensorised Gelfand-Shilov spaces

In the following we will use the following notation. Let $s, t \in \mathbb{N}$, s. t. $s + t = d$. Therefore we write for $x \in \mathbb{R}^{s+t} = \mathbb{R}^d$: $x = (x^1, x^2) = (x_1, \ldots, x_s, x_{s+1} \ldots, x_{s+t})$ and similarly for $p \in \mathbb{N}_0^{s+t}$: $p = (p^1, p^2) = (p_1, \ldots, p_s, p_{s+1} \ldots, p_{s+t})$. In addition we put $m, k, q \in \mathbb{N}_0^d$ accordingly.

As a consequence of inequalities proved by Kahspirovskij, Pilipovic and others we have the following proposition.

Proposition 3.1. *Let $x \in \mathbb{R}^d$ and $k, m \in \mathbb{N}_0^d$*

1.
$$\begin{aligned} x^m\Psi_k(x) &= \prod_{i=1}^{d} x_i^{m_i}\psi_{k_i}(x_i) \\ &= \prod_{i=1}^{d-1} 2^{-m_d} x_i^{m_i} \sum_{r_d=0}^{m_d} \psi_{k_i}(x_i) c^{k_d}_{r_d,m_d}\psi_{k_d-m_d+2+2r_d}(x_d) \\ &= 2^{-m} \sum_{\substack{r_i=0,\\ i=1,\dots,d}}^{m_i} \prod_{i=0}^{d} c^{k_i}_{r_i,m_i}\psi_{k_i-m_i+2+2r_i}(x_i) \end{aligned}$$

where, for $j = 1,\dots,d$
$$\left|c^{k_j}_{r_j,m_j}\right| \le \binom{m_j}{r_j}[(2k_j+1)^{m_j/2} + m_j^{m_j/2}].$$

2. *Put* $\mathcal{R}_i^{j_i} = (x_i^2 - \partial_i^2)^{j_i}, i = 1,\cdots,d$. *Then for* $f \in C^\infty(\mathbb{R}^d)$,
$$\begin{aligned} \mathcal{R}_1^{j_1}\dots\mathcal{R}_d^{j_d} f &= \mathcal{R}_1^{j_1}\dots\mathcal{R}_{d-1}^{j_{d-1}} \sum_{\substack{p_d+q_d=2k_d,\\ k_d\le j_d}} C^{j_d}_{p_d,q_d} x_d^{p_d}\partial^{q_d} f(x_1,\dots,x_d) \\ &= \sum_{\substack{p_i+q_i=2k_i, k_i\le j_i,\\ i=1,\dots,d}} \prod_{i=1}^{d} C^{j_i}_{p_i,q_i} x_i^{p_i}\partial^{q_i} f(x_1,\dots,x_d), \end{aligned}$$

where for $i = 1,\dots,d$
$$\left|C^{j_i}_{p_i,q_i}\right| \le 10^{j_i} j_i^{j_i - \frac{p_i+q_i}{2}}.$$

Definition 3.2. We define the tensored spaces $\big(\mathcal{S}^\alpha_\alpha\otimes\mathcal{S}^\beta_\beta\big)(\mathbb{R}^{s+t})$ and $\big(\Sigma^\alpha_\alpha\otimes\Sigma^\beta_\beta\big)(\mathbb{R}^{s+t})$ as follows:
$$\big(\mathcal{S}^\alpha_\alpha\otimes\mathcal{S}^\beta_\beta\big)(\mathbb{R}^{s+t}) := \Big\{ f\in\mathbb{S}(\mathbb{R}^d) \ : \exists\ A,B,C \text{ s.t. } \forall p,q \quad \|x^{1p_1}x^{2p_2}\partial^{q^1}_{x^1}\partial^{q^2}_{x^2} f\|_{L^2(\mathbb{R}^d)} \le CA^{|p|}p^1!^\alpha q^1!^\alpha B^{|q|}p^2!^\beta q^2!^\beta \Big\}$$
$$\big(\Sigma^\alpha_\alpha\otimes\Sigma^\beta_\beta\big)(\mathbb{R}^{s+t}) := \Big\{ f\in\mathbb{S}(\mathbb{R}^d) \ : \exists C\ \forall h \text{ s.t. } \forall p,q \quad \|x^{1p_1}x^{2p_2}\partial^{q^1}_{x^1}\partial^{q^2}_{x^2} f\|_{L^2(\mathbb{R}^d)} \le Ch^{|p|+|q|}p^1!^\alpha q^1!^\alpha p^2!^\beta q^2!^\beta \Big\}$$

The next proposition is formulated in the simple form so that it can be used for the characterization of new Gelfand-Shilov type spaces.

Theorem 3.3. *Let* $1/2 \le \nu < \mu$ *and* $s,t\in\mathbb{N}_0$ *such that* $s+t=d$.

1. *Let* $f\in C^\infty(\mathbb{R}^{s+t})$. *If for some* $A>0$ *and some* $C>0$ *(resp. for every* $A>0$ *there exists* $C>0$*) such that*
$$\|x^{1p^1}x^{2p^2}\partial^{q^1}_{x^1}\partial^{q^2}_{x^2} f(x^1,x^2)\|_{L^2} \le CA^{|p||q|}p^1!^\nu p^2!^\mu q^1!^\nu q^2!^\mu, \tag{3.1}$$

then $f \in L^2(\mathbb{R}^{s+t})$, $f(x) = \sum_{k\in\mathbb{N}_0^{s+t}} a_{(k^1,k^2)}\psi_{k^1}(x^1)\psi_{k^2}(x^2)$, $x \in \mathbb{R}^{s+t}$, *and there exist constants* $C > 0$ *and* $\delta > 0$ *(respectively, for every* $\delta > 0$ *there exists a* $C > 0$*) such that*

$$|a_k| \le C \exp\left(-\delta\left(|k^1|^{1/(2\nu)} + |k^2|^{1/(2\mu)}\right)\right),\ k \in \mathbb{N}_0^{s+t}. \tag{3.2}$$

2. *If* $f \in L^2(\mathbb{R}^{s+t})$, $f(x) = \sum_{k\in\mathbb{N}_0^{s+t}} a_{(k^1,k^2)}\psi_{k^1}(x^1)\psi_{k^2}(x^2), x \in \mathbb{R}^{s+t}$, *satisfies* (3.2) *for some* $C > 0$ *and* $\delta > 0$ *(respectively, for every* $\delta > 0$ *there exists* $C > 0$*), then* $f \in C^\infty(\mathbb{R}^{s+t})$ *and* (3.1) *holds with some* $A > 0$ *and* $C > 0$ *(respectively, for every* $A > 0$ *there exists* $C > 0$*).*

Proof. Assume (3.1) to hold as well as that $j = (j^1, j^2) \in \mathbb{N}_0^{s+t}$. We have

$$\mathcal{R}_1^{j_1}\dots\mathcal{R}_d^{j_d} f = \sum_{k\in\mathbb{N}_0^d} a_k \mathcal{R}_1^{j_1}\dots\mathcal{R}_d^{j_d}\psi_{k_1}\cdot\dots\cdot\psi_{k_d} = \sum_{k\in\mathbb{N}_0^d} a_k \prod_{i=1}^d (2k_i+1)^{j_i}\psi_{k_i}.$$

There exists a constant $\mathcal{C} > 0$ such that

$$\begin{aligned}
\|\mathcal{R}_1{}^{j_1}\dots\mathcal{R}_d{}^{j_d} f\|_{L^2} &= \sum_{\substack{p_i+q_i=2k_i, k_i\le j_i,\\ i=1,\dots,d}} \prod_{i=1}^d C^{j_i}_{p_i,q_i}\|x_i^{p_i}\partial^{q_i} f\|_{L^2(\mathbb{R}^d)}\\
&\le \mathcal{C}\,10^j \sum_{\substack{p_i+q_i=2k_i, k_i\le j_i,\\ i=1,\dots,d}} \prod_{i=1}^d j_i^{j_i-k_i} p^1!^\nu q^1!^\nu p^2!^\mu q^2!^\mu l^k\\
&\le \mathcal{C}\,10^j \sum_{\substack{p_i+q_i=2k_i, k_i\le j_i,\\ i=1,\dots,d}} \prod_{i=1}^d j_i^{j_i-k_i} k^1!^{2\nu} k^2!^{2\mu} (2lr)^k \frac{1}{2^k}\\
&\le \mathcal{C}\,(20lr)^j \sum_{\substack{p_i+q_i=2k_i, k_i\le j_i,\\ i=1,\dots,d}} \prod_{i=1}^d j_i^{j_i-k_i} \frac{k^1!^{2\nu} k^2!^{2\mu}}{j^1!2\nu j^2!^{2\mu}} j^1!^{2\nu} j^2!^{2\mu}\frac{1}{2^k}.
\end{aligned}$$

This implies, using

$$\left(\sum_{k\in\mathbb{N}_0^d} |a_k|^2 \prod_{i=1}^d (2k_i+1)^{2j_i}\psi_{k_i}\right)^{1/2} = \|\mathcal{R}_1{}^{j_1}\dots\mathcal{R}_d{}^{j_d} f\|_{L^2}$$

and with new constants

$$|a_k|\prod_{i=1}^d (2k_i+1)^{j_i} \le \mathcal{C}c^j j^1!^{2\nu} j^2!^{2\mu}.$$

Thus, with suitable $C > 0$ and $\delta > 0$,

$$|a_k| \le Ce^{-\delta(|k^1|^{1/(2\nu)}+|k^2|^{1/(2\mu)})}.$$

b) Now, assume (3.2) and let $m = (m^1, m^2) \in \mathbb{N}_0^{s+t}$. We have

$$\|{x^1}^{m^1} {x^2}^{m^2} f\|_{L^2} = \|\sum_{k\in\mathbb{N}_0^d} a_k {x^1}^{m^1} {x^2}^{m^2} \psi_{k^1}\psi_{k^2}\|_{L^2}$$
$$\leq 2^{-m^1/2-m^2/2}.$$

Therefore we get with $\binom{m}{k} = \prod_{i=1}^d \binom{m_i}{k_i}$

$$\sum_{k\in\mathbb{N}_0^d} |a_k| \left(\sum_{\substack{j_i\leq m_i,\\ i=1,\dots,d}} \binom{m}{j} \left[(2k_1+1)^{m_1/2} + m_1^{m_1/2}\right]\cdot\ldots\cdot\left[(2k_d+1)^{m_d/2} + m_d^{m_d/2}\right] \right)$$
$$\leq 2^{m/2} \left(\sum_{k\in\mathbb{N}_0^d} |a_k|^2 C_k^2\right)^{1/2} \left(\sum_{k\in\mathbb{N}_0^d} C_k^{-2}\bar{C}_k^2\right)^{1/2},$$

where $C_k = \exp\left(\delta\left({k^1}^{1/(2\nu)} + {k^2}^{1/(2\mu)}\right)\right)$ and $\bar{C}_k = \prod_{i=1}^d [(2k_i+1)^{m_i/2} + m_i^{m_i/2}]$.

There exists a constant $\mathcal{C}$ such that

$$\|{x^1}^{m^1} {x^2}^{m^2} f\|_{L^2} \leq \mathcal{C}\, 2^{\frac{m}{2}} \left[\sum_{k\mathbb{N}_0^d} C_k^{-2}\bar{C}_k^2\right]^{1/2}$$
$$\leq \mathcal{C}\, 2^{\frac{m}{2}}\, \bar{C} \left[\sum_{k\in\mathbb{N}_0^d} \exp\left(-\delta\left(\left(2k^1+1\right)^{1/(2\nu)} + \left(2k^2+1\right)^{1/(2\mu)}\right)\right)\right]^{\frac{1}{2}}$$
$$\leq \mathcal{C}\, 2^{\frac{m}{2}}\, m^1!^\nu m^2!^\mu\, \tilde{C} + \frac{{m^1}^{m^1/2} {m^2}^{m^2/2}}{m^1!^\nu m^2!^\mu} m^1!^\nu m^2!^\mu,$$

where $\bar{C} = \prod_{i=1}^d C_i$ and

$$\bar{C}_i := \begin{cases} \sup_{k_i\in\mathbb{N}_0} ((2k_i+1)^{m_i/2} + {m_i}^{m_i/2}) e^{-\frac{1}{2}\delta(2k_i+1)^{1/(2\nu)}} & \text{for } i = 1,\dots,s \\ \sup_{k_i\in\mathbb{N}_0} ((2k_i+1)^{m_i/2} + {m_i}^{m_i/2}) e^{-\frac{1}{2}\delta(2k_i+1)^{1/(2\mu)}} & \text{for } i = s+1,\dots,t \end{cases}$$

and $\tilde{C} = \prod_{i=1}^d \tilde{C}_i$ with

$$\tilde{C}_i := \begin{cases} \sup_{k_i\in\mathbb{N}_0} \frac{(2k_i+1)^{m_i/2} e^{-\frac{1}{2}\delta(2k_i+1)^{1/(2\nu)}}}{m_i!^\nu} & \text{for } i = 1,\dots,s \\ \sup_{k_i\in\mathbb{N}_0} \frac{(2k_i+1)^{m_i/2} e^{-\frac{1}{2}\delta(2k_i+1)^{1/(2\mu)}}}{m_i!^\mu} & \text{for } i = s+1,\dots,t \end{cases}$$

We conclude that

$$\|{x^1}^{m^1} {x^2}^{m^2} f\|_{L^2} \leq \mathcal{C}\, 2^{m/2} m^1!^\nu m^2!^\mu. \tag{3.3}$$

By the Fourier transformation, we have

$$\|\partial_{x^1}^{m^1} \partial_{x^2}^{m^2} f\|_{L^2} \leq \mathcal{C}\, 2^{m/2} m^1!^\nu m^2!^\mu. \tag{3.4}$$

Let $p, q \in \mathbb{N}_0^d$. Then

$$
\begin{aligned}
\|x^p\partial^q f\|_{L^2}^2 &= (x^p\partial^q f, x^p\partial^q f)_{L^2} = \left|\left(\partial^q\left(x^{2p}\partial^q f\right), f\right)_{L^2}\right| \\
&\le \left|\sum_{\substack{\kappa\in\mathbb{N}_0^d\\ \kappa_i\le\gamma_i}} \binom{q}{\kappa}\frac{(2p)!}{(2p-\kappa)!}\,(x^{2p-\kappa}f^{(2q-\kappa)}, f)_{L^2}\right| \\
&\le \sum_{\substack{\kappa\in\mathbb{N}_0^d\\ \kappa_i\le\gamma_i}} \binom{q}{\kappa}\binom{2p}{\kappa}\kappa!\,\|x^{2p-\kappa}f\|_{L^2}\,\|\partial^{2q-\kappa}f\|_{L^2},
\end{aligned}
$$

where $\gamma_i := \min\{q_i, 2p_i\}, i = 1, \ldots, d$.

This implies that (3.1) holds. □

Corollary 3.4. Let $s, t \in \mathbb{N}_0^d$ s.t. $s + t = d$, and $1/2 \le \alpha < \beta$ in the case $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $1/2 < \alpha < \beta$ in the case $\sum_\alpha^\beta(\mathbb{R}^d)$) then it holds

$\Sigma_\alpha^\alpha(\mathbb{R}^d) \subset \left(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta\right)(\mathbb{R}^{s+t}) \subset \Sigma_\beta^\beta(\mathbb{R}^d)$, $\Sigma_\alpha^\alpha(\mathbb{R}^d) \subset \left(\Sigma_\beta^\beta \otimes \Sigma_\alpha^\alpha\right)(\mathbb{R}^{s+t}) \subset \Sigma_\beta^\beta(\mathbb{R}^d)$,

$\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \subset \left(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta\right)(\mathbb{R}^{s+t}) \subset \mathcal{S}_\beta^\beta(\mathbb{R}^d)$ and $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \subset \left(\mathcal{S}_\beta^\beta \otimes \mathcal{S}_\alpha^\alpha\right)(\mathbb{R}^{s+t}) \subset \mathcal{S}_\beta^\beta(\mathbb{R}^d)$.

The inclusions are strict and continuous.

Proof. The inclusions are obviously continuous. The statement that the inclusions are strict follows straightforward from Theorems 2.1 and 3.3. □

Moreover it is clear that $\Sigma_\alpha^\beta(\mathbb{R}^{s+t})$ is not a subset of $\left(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta\right)(\mathbb{R}^{s+t})$ and that the opposite inclusion also does not hold; the same is true for $\mathcal{S}_\alpha^\beta(\mathbb{R}^{s+t})$ and $\left(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta\right)(\mathbb{R}^{s+t})$.

4. Gelfand–Shilov type spaces related to the tensorised harmonic oscillators on $\mathbb{R}^d$

We introduce one more class of Gevrey Gelfand-Shilov-type spaces.

$$
S_\sigma^{\otimes,\sigma}(\mathbb{R}^d) = \mathrm{indlim}_{\delta\to 0} S_\sigma^{\otimes,\sigma}(\mathbb{R}^d;\delta) \text{ and } \Sigma_\sigma^{\otimes,\sigma} = \mathrm{projlim}_{\delta\to\infty} S_\sigma^{\otimes,\sigma}(\mathbb{R}^d;\delta)
$$

where

$$
\begin{aligned}
S_\sigma^{\otimes,\sigma}(\mathbb{R}^d;\delta) &= \{f \in \mathbb{S}(\mathbb{R}^d) : \|f\|_{\otimes;\sigma,\delta} < \infty\} \text{ with} \\
\|f\|_{\otimes;\sigma,\delta} &= \sum_{m\in\mathbb{N}_0^d} |a_m|^2 \exp\left(2\delta((m_1+1)\ldots(m_d+1))^{1/(2\sigma d)}\right)
\end{aligned}
$$

where the $\left(a_m\right)_{m\in\mathbb{N}_0^d}$ are the Hermite coefficients of f (cf. (1.1)).

Let

$$
\vec{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d, \quad b_j > -1,\ j = 1, \ldots, d. \tag{4.1}
$$

Define

$$\mathcal{H}_{\vec{b}} = (-\partial_{x_1}^2 + x_1^2 + b_1)\ldots(-\partial_{x_d}^2 + x_d^2 + b_d) = \otimes_{j=1}^d(-\partial_{x_j}^2 + x_j^2 + b_j) \quad (4.2)$$

By the arguments as in Theorem 3.3 one can prove the next theorem.

Theorem 4.1. *Under the hypotheses given above the following conditions are equivalent:*

1. $f \in \mathcal{S}_\sigma^{\otimes,\sigma}(\mathbb{R}^d)$, $\sigma \geq 1/2$, *resp.*, $f \in \Sigma_\sigma^{\otimes,\sigma}(\mathbb{R}^d)$, $\sigma > 1/2$.
2. *there exists* $A > 0$, *resp., for every* $A > 0$,
$$\left\|\mathcal{H}_{\vec{b}}^r f\right\| \leq A^{r+1} r!^{2d\sigma}, \quad r \in \mathbb{N}_0,$$
where $\sigma \geq 1/2$, *resp.*, $\sigma > 1/2$.
3. *if* $f = \sum_{k=0}^{\infty} a_k \Psi_k(x)$, *there exist* $C, \varepsilon > 0$, *resp. for every* $\varepsilon > 0$ *there exists* $C > 0$ *such that*
$$|a_k| \leq C \exp\big(-\varepsilon(k \log^{-(d-1)}(1+k))^{1/(2d\sigma)}\big),$$
where $\sigma \geq 1/2$, *resp.*, $\sigma > 1/2$.

Example 4.2. With $\sigma = 1/2$ we obtain $\phi = \sum_{k\in\mathbb{N}_0^d} a_k \Psi_k \in \mathcal{S}_{1/2}^{\otimes,1/2}(\mathbb{R}^d)$, if and only if $|a_k| \leq C \exp\big(-\delta k^{1/d}(\log(1+k))^{-1+1/d}\big)$, $k \in \mathbb{N}_0$, for some $C > 0, \delta > 0$.

It is natural to see the relations of the space $\mathcal{S}_\sigma^{\otimes,\sigma}(\mathbb{R}^d)$, resp., $\Sigma_\sigma^{\otimes,\sigma}(\mathbb{R}^d)$ and $S_\sigma^\sigma(\mathbb{R}^d)$, resp., $\Sigma_\sigma^\sigma(\mathbb{R}^d)$ $\sigma > 0$, $\sigma \geq 1/2$, resp., $\sigma > 1/2$.

Since

$$((2k_1+1)...(2k_d+1))^{1/d} \leq 2(k_1 + ... + k_d + 1),$$

it follows

Theorem 4.3. *Let* $d \geq 2$ *and* $\sigma \geq 1/2$, *resp.* $\sigma > 1/2$. *Then*

$$\mathcal{S}_\sigma^\sigma(\mathbb{R}^d) \hookrightarrow \mathcal{S}_\sigma^{\otimes,\sigma}(\mathbb{R}^d) \hookrightarrow \mathcal{S}_{2\sigma}^{2\sigma}(\mathbb{R}^d),$$

resp.,

$$\Sigma_\sigma^\sigma(\mathbb{R}^d) \hookrightarrow \Sigma_\sigma^{\otimes,\sigma}(\mathbb{R}^d) \hookrightarrow \Sigma_{2\sigma}^{2\sigma}(\mathbb{R}^d).$$

The inclussions are strict.

Remark 4.4. The relation with the Gelfan-Shilov tensor-type spaces are clear.

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